

# Asymptotic late-time tails of massive spin-2 fields

Shahar Hod

*The Ruppin Academic Center, Emeq Hefer 40250, Israel*  
and

*The Hadassah Institute, Jerusalem 91010, Israel*

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The late-time dynamics of massive spin-2 fields in flat and curved spacetimes are studied analytically. We find that the time evolutions of the massive fields are characterized by oscillatory power-law decaying tails at asymptotically late times. In a flat spacetime the decaying exponent depends on the multipole number and the parity of the mode. In the curved Schwarzschild black-hole spacetime the decaying exponent is found to be universal.

## I. INTRODUCTION

Massive bosonic fields with arbitrary spins are predicted by many beyond-standard-model theories [1–5]. In addition, several extensions of the general theory of relativity which include massive mediator fields have been proposed in the literature, see [1–5] and references therein. It is therefore of physical interest to explore the characteristic dynamics of these massive fields.

One of the most remarkable features of the dynamics of massive fields is the development of asymptotic ( $t \rightarrow \infty$ ) tails. At late times, after the passage of the primary pulse, massive wave fields do not cut off sharply. Instead, they tend to die off gently, leaving behind asymptotically late-time decaying tails [6–8]. Massive fields are unique in that their *flat*-space dynamics are characterized by non-vanishing tails. Massless fields, on the other hand, are characterized by tail-free propagation in a flat spacetime. [For the well-studied phenomena of massless wave tails in curved spacetimes, see [9] and references therein.]

The asymptotic tails which characterize the late-time dynamics of massive fields are a direct consequence of the fact that different frequencies which compose the massive wave packet have different phase velocities. For example, the flat-space dynamics of a spherically symmetric spin-0 (scalar) field of mass  $\mu$  is governed by the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + \mu^2\right)\Psi_0(t, r) = 0. \quad (1)$$

[We use units in which  $G = c = \hbar = 1$ . In these units  $\mu$  has the dimensions of  $(\text{length})^{-1}$ .] For a plane wave of the form  $\Psi(t, r) \sim e^{i(kr - \omega t)}$ , one finds from (1) the dispersion relation  $\omega(k) = \sqrt{k^2 + \mu^2}$  which yields the frequency-dependent phase velocity [6]

$$v_{\text{phase}} = [1 - (\mu/\omega)^2]^{-1/2}. \quad (2)$$

For a spherically symmetric massive scalar field, it was shown in [6] that the interference between the different components of the massive wave packet [which are characterized by different phase velocities, see Eq. (2)] produces a late-time decaying tail of the form [6]

$$\Psi_0(t \gg r) \sim t^{-3/2} \cos(\mu t). \quad (3)$$

Here  $r$  is the location of the observer.

The late-time tails of generic (non-spherical) massive scalar fields were analyzed in [7]. Resolving the field into spherical harmonics  $\Psi(r, t) = \sum_{lm} \psi_{lm}(r) Y_{lm}(\theta, \phi) e^{-i\omega t}$  [10], one obtains the flat-space Schrödinger-like wave equation [7, 11]:

$$\left[\frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{l(l+1)}{r^2}\right]\psi_l(r) = 0, \quad (4)$$

where the multipole number  $l$  is a non-negative integer. It was shown in [7] that the asymptotic late-time dynamics of the scalar field (4) is characterized by an oscillatory decaying tail of the form

$$\Psi_l(t \gg r) \sim t^{-(l+\frac{3}{2})} \cos(\mu t). \quad (5)$$

It is worth pointing out that the oscillatory term  $\cos(\mu t)$  in Eqs. (3) and (5) reflects the fact that the late-time ( $t \rightarrow \infty$ ) dynamics of a massive field is dominated by only a small fraction of the frequencies which compose the wave packet, those with

$$|\omega| = \mu - O(t^{-1}) , \quad (6)$$

see [6, 7] for details.

In the present paper we shall analyze the late-time tails of massive spin-2 fields. The wave equations for these massive fields were derived most recently in [5]. We shall first consider the time evolution of the massive fields in a flat spacetime. We shall then analyze the late-time dynamics in the curved Schwarzschild black-hole spacetime.

The problem of analyzing the late-time tails of higher-spin ( $s > 0$ ) massive fields is quite challenging. The main technical problem arises due to the coupling between the different degrees of freedom which characterize a higher-spin massive field [4, 5] (this coupling is due to the broken gauge invariance and the additional longitudinal degree of freedom which characterizes the dynamics of massive fields [4, 5, 12]). As a consequence, the dynamics of higher-spin massive fields are governed by systems of *coupled* differential equations [13]. We shall show, however, that the wave equations can be decoupled in the asymptotic regime  $r \gg \mu^{-1}$ . This fact will allow us to study analytically the asymptotic late-time dynamics of the massive fields.

## II. THE ASYMPTOTIC LATE-TIME TAILS IN A FLAT SPACETIME

We shall now analyze the asymptotic late-time tails of massive spin-2 fields. The various field modes are classified as even-parity modes or odd-parity modes according to their behavior under a parity inversion transformation of the form  $\theta \rightarrow \pi - \theta$  and  $\phi \rightarrow \pi + \phi$  [4, 5]. [Here  $\theta$  and  $\phi$  are the polar and azimuthal angles, respectively.] Even-parity modes are multiplied by  $(-1)^l$  under this inversion transformation, whereas odd-parity modes are multiplied by  $(-1)^{l+1}$ . We shall show below that the dynamics of these two parity modes are characterized by different late-time asymptotic behaviors.

### A. The odd-parity dipole ( $l = 1$ ) mode

The dynamics of the odd-parity dipole ( $l = 1$ ) mode [14] of massive spin-2 fields is governed by a single Schrödinger-like wave equation of the form [5, 15]

$$\left( \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{6}{r^2} \right) \psi(r) = 0 . \quad (7)$$

Note that Eq. (7) is of the form (4) with  $l_{\text{eff}} = 2$  as the effective multipole parameter. Thus, taking cognizance of Eq. (5), one finds that the asymptotic late-time tail of the odd-parity dipole ( $l = 1$ ) mode is given by

$$\Psi(t \gg r) \sim t^{-7/2} \cos(\mu t) . \quad (8)$$

### B. Generic ( $l \geq 2$ ) odd-parity modes

The dynamics of generic ( $l \geq 2$ ) odd-parity modes of massive spin-2 fields is governed by a system of two coupled ordinary differential equations [5, 16]:

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{l(l+1)+4}{r^2} \right] \psi_1(r) = \frac{2[l(l+1)-2]}{r^2} \psi_2(r) \quad (9)$$

and

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{l(l+1)-2}{r^2} \right] \psi_2(r) = \frac{2}{r^2} \psi_1(r) . \quad (10)$$

Defining [17]

$$\phi_{\pm} \equiv \psi_1 + \frac{-3 \pm (2l+1)}{2} \psi_2 , \quad (11)$$

one obtains a pair of *decoupled* wave equations:

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{(l+1)(l+2)}{r^2} \right] \phi_+(r) = 0 \quad (12)$$

and

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{l(l-1)}{r^2} \right] \phi_-(r) = 0 . \quad (13)$$

Note that Eq. (12) is of the form (4) with  $l_{\text{eff}} = l+1$  as the effective multipole parameter. Thus, taking cognizance of Eq. (5), one finds that the asymptotic late-time tail associated with the field  $\phi_+$  is given by

$$\Phi_+(t \gg r) \sim t^{-(l+\frac{5}{2})} \cos(\mu t) . \quad (14)$$

Likewise, note that Eq. (13) is of the form (4) with  $l_{\text{eff}} = l-1$  as the effective multipole parameter. Thus, taking cognizance of Eq. (5), one finds that the asymptotic late-time tail associated with the field  $\phi_-$  is given by

$$\Phi_-(t \gg r) \sim t^{-(l+\frac{1}{2})} \cos(\mu t) . \quad (15)$$

### C. The even-parity monopole ( $l=0$ ) mode

The dynamics of the even-parity monopole mode of massive spin-2 fields is governed by a single Schrödinger-like wave equation of the form [5, 18]

$$\left( \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{6}{r^2} \right) \psi(r) = 0 . \quad (16)$$

Note that Eq. (16) is of the form (4) with  $l_{\text{eff}} = 2$  as the effective multipole parameter [19]. Thus, taking cognizance of Eq. (5), one finds that the asymptotic late-time tail of the even-parity monopole ( $l=0$ ) mode is given by

$$\Psi(t \gg r) \sim t^{-7/2} \cos(\mu t) . \quad (17)$$

### D. The even-parity dipole ( $l=1$ ) mode

The dynamics of the even-parity dipole mode of massive spin-2 fields is governed by a system of two coupled ordinary differential equations [5]. These differential equations are rather cumbersome [see Eqs. (44)-(45) of [5]] but after some tedious algebra one finds that, in the asymptotic limit  $r \gg \mu^{-1}$ , the two equations can be simplified to yield the following system of coupled differential equations [20, 21]:

$$\left( \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{8}{r^2} \right) \psi_1(r) = -\frac{4}{r^2} \psi_2(r) \quad (18)$$

and

$$\left( \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{6}{r^2} \right) \psi_2(r) = -\frac{6}{r^2} \psi_1(r) . \quad (19)$$

Defining [22]

$$\phi_+ \equiv \psi_1 + \psi_2 \quad \text{and} \quad \phi_- \equiv \psi_1 - \frac{2}{3} \psi_2 , \quad (20)$$

one obtains a pair of *decoupled* wave equations:

$$\left( \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{2}{r^2} \right) \phi_+(r) = 0 \quad (21)$$

and

$$\left( \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{12}{r^2} \right) \phi_-(r) = 0 . \quad (22)$$

Note that Eq. (21) is of the form (4) with  $l_{\text{eff}} = 1$  as the effective multipole parameter. Thus, taking cognizance of Eq. (5), one finds that the asymptotic late-time tail associated with the field  $\phi_+$  is given by

$$\Phi_+(t \gg r \gg \mu^{-1}) \sim t^{-5/2} \cos(\mu t) . \quad (23)$$

Likewise, note that Eq. (22) is of the form (4) with  $l_{\text{eff}} = 3$  as the effective multipole parameter. Thus, taking cognizance of Eq. (5), one finds that the asymptotic late-time tail associated with the field  $\phi_-$  is given by

$$\Phi_-(t \gg r \gg \mu^{-1}) \sim t^{-9/2} \cos(\mu t) . \quad (24)$$

### E. Generic ( $l \geq 2$ ) even-parity modes

The dynamics of generic ( $l \geq 2$ ) even-parity modes of massive spin-2 fields is governed by a system of three coupled ordinary differential equations [5]. These differential equations are rather lengthy [see Eqs. (38)-(40) of [5]] but after some tedious algebra one finds that, in the asymptotic limit  $r \gg \mu^{-1}$ , the three equations can be simplified to yield the following system of coupled differential equations [20, 23]:

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{l(l+1)+6}{r^2} \right] \psi_1(r) = -\frac{2l(l+1)}{r^2} \psi_2(r) , \quad (25)$$

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{l(l+1)+4}{r^2} \right] \psi_2(r) = -\frac{6}{r^2} \psi_1(r) + \frac{2[l(l+1)-2]}{r^2} \psi_3(r) , \quad (26)$$

and

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{l(l+1)-2}{r^2} \right] \psi_3(r) = \frac{2}{r^2} \psi_2(r) . \quad (27)$$

Defining [24]

$$\phi_- \equiv \psi_1 + \frac{2}{3}(l+1)\psi_2 - \frac{(l+1)[l(l+1)-2]}{3(l-1)}\psi_3 , \quad (28)$$

$$\phi_+ \equiv \psi_1 - \frac{2}{3}l\psi_2 - \frac{l[l(l+1)-2]}{3(l+2)}\psi_3 , \quad (29)$$

and

$$\phi_0 \equiv \psi_1 + \psi_2 + [l(l+1)-2]\psi_3 , \quad (30)$$

one obtains a system of three *decoupled* wave equations:

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{(l-1)(l-2)}{r^2} \right] \phi_-(r) = 0 , \quad (31)$$

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{(l+2)(l+3)}{r^2} \right] \phi_+(r) = 0 , \quad (32)$$

and

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{l(l+1)}{r^2} \right] \phi_0(r) = 0 \quad (33)$$

Note that Eq. (31) is of the form (4) with  $l_{\text{eff}} = l-2$  as the effective multipole parameter. Thus, taking cognizance of Eq. (5), one finds that the asymptotic late-time tail associated with the field  $\phi_-$  is given by

$$\Phi_-(t \gg r \gg \mu^{-1}) \sim t^{-(l-\frac{1}{2})} \cos(\mu t) . \quad (34)$$

Likewise, note that Eq. (32) is of the form (4) with  $l_{\text{eff}} = l+2$  as the effective multipole parameter. Thus, taking cognizance of Eq. (5), one finds that the asymptotic late-time tail associated with the field  $\phi_+$  is given by

$$\Phi_+(t \gg r \gg \mu^{-1}) \sim t^{-(l+\frac{3}{2})} \cos(\mu t) . \quad (35)$$

Finally, note that Eq. (33) is of the form (4) with  $l_{\text{eff}} = l$  as the effective multipole parameter. Thus, taking cognizance of Eq. (5), one finds that the asymptotic late-time tail associated with the field  $\phi_0$  is given by

$$\Phi_0(t \gg r \gg \mu^{-1}) \sim t^{-(l+\frac{3}{2})} \cos(\mu t) . \quad (36)$$

### III. THE ASYMPTOTIC LATE-TIME TAIL IN THE CURVED SCHWARZSCHILD SPACETIME

The wave equations for the various modes of massive spin-2 fields in the Schwarzschild black-hole spacetime are rather lengthy, see Eq. (36) of [5] for the odd-parity dipole ( $l = 1$ ) mode, Eqs. (32)-(35) of [5] for generic ( $l \geq 2$ ) odd-parity modes, Eq. (30) of [5] for the even-parity monopole ( $l = 0$ ) mode, Eqs. (44)-(45) of [5] for the even-parity dipole ( $l = 1$ ) mode, and Eqs. (38)-(40) of [5] for generic ( $l \geq 2$ ) even-parity modes. Remarkably, after some algebra one finds that the wave equations of the various modes are all characterized by the *same* asymptotic behavior:

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 + \frac{2M\mu^2}{r} + O\left(\frac{1}{r^2}\right) \right] \psi(r) = 0 \quad \text{for } r \gg \max\{1/M\mu^2, \mu^{-1}\}, \quad (37)$$

where  $M$  is the mass of the central black hole. It is important to note that the new term  $2M\mu^2/r$ , which represents the curvature of the black-hole spacetime [25], dominates over the coupling terms in the asymptotic limit  $r \gg M/(M\mu)^2$  [26].

It was shown in [27] that the asymptotic late-time dynamics of a wave field of the form (37) is characterized by an oscillatory decaying tail of the form

$$\Psi(t \gg r \gg 1/M\mu^2) \sim t^{-5/6} \cos(\mu t). \quad (38)$$

Thus, the late-time evolution of massive fields in the curved Schwarzschild black-hole spacetime is universal in the sense that all modes share the same asymptotic behavior (38) [28]. It is worth emphasizing that, for  $M\mu \ll 1$ , the flat-space tails of Sec. II dominate the dynamics of the massive fields in the intermediate asymptotic regime  $M \ll t \ll M/(M\mu)^2$ , see [7] for details.

### IV. SUMMARY

The characteristic late-time tails of massive spin-2 fields in flat and curved spacetimes were studied analytically. The problem of analyzing the dynamics of higher-spin ( $s > 0$ ) massive fields is technically more challenging than the corresponding problem of analyzing the time evolution of massless fields. The main technical difficulty stems from the fact that different degrees of freedom which characterize the dynamics of massive fields are coupled [4, 5]. As a consequence, the dynamics of higher-spin massive fields are governed by systems of *coupled* ordinary differential equations [13]. We have shown, however, that the wave equations can be decoupled in the asymptotic regime  $r \gg \mu^{-1}$ . This fact allows one to study analytically the properties of the asymptotic tails which characterize the late-time dynamics of the massive fields. The various cases studied and the corresponding asymptotic late-time tails are summarized in Table I.

Mode(s)	$l_{\text{eff}}$	Asymptotic late-time tail(s)
Odd-parity $l = 1$	2	$t^{-\frac{7}{2}} \cos(\mu t)$
Odd-parity $l \geq 2$	$l - 1$	$t^{-(l+\frac{1}{2})} \cos(\mu t)$
	$l + 1$	$t^{-(l+\frac{5}{2})} \cos(\mu t)$
Even-parity $l = 0$	2	$t^{-\frac{7}{2}} \cos(\mu t)$
Even-parity $l = 1$	1	$t^{-\frac{9}{2}} \cos(\mu t)$
	3	$t^{-\frac{9}{2}} \cos(\mu t)$
Even-parity $l \geq 2$	$l - 2$	$t^{-(l-\frac{1}{2})} \cos(\mu t)$
	$l$	$t^{-(l+\frac{3}{2})} \cos(\mu t)$
	$l + 2$	$t^{-(l+\frac{7}{2})} \cos(\mu t)$

TABLE I: Late time tails of massive spin-2 fields in a flat spacetime. Here  $l$  is the multipole number of the mode and  $l_{\text{eff}}$  is the effective multipole parameter which determines the asymptotic behavior of the scattering potential in the  $r \rightarrow \infty$  limit [see Eq. (4)]. The asymptotic late time tail in the curved Schwarzschild black-hole spacetime is given by  $\Psi(t \gg r) \sim t^{-5/6} \cos(\mu t)$  for all modes [28].

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  - [10] Throughout the paper we shall use the capital Greek letters  $\Psi(t, r)$  and  $\Phi(t, r)$  to denote the time-dependent fields and the minuscules Greek letters  $\psi(r; \omega)$  and  $\phi(r; \omega)$  to denote the radial parts of the fields in the frequency domain.
  - [11] It is worth emphasizing that the wave equations considered in the present work have a quadratic dependence of the effective potentials on the frequency, whereas in the standard Schrödinger case there is a linear dependence on the energy.
  - [12] It is worth emphasizing that higher spin ( $s > 0$ ) massive fields have longitudinal radiative degrees of freedom which do not exist in the spectra of their massless counterparts. In particular, massive higher spin fields have low multipoles propagating modes with  $l < s$ , for example: the vector ( $s = 1$ ) field monopole ( $l = 0$ ) and the graviton ( $s = 2$ ) field monopole ( $l = 0$ ) and dipole ( $l = 1$ ) modes (we recall that the radiative modes of massless fields are characterized by the inequality  $l \geq s$ ).
  - [13] The only exceptions are the odd-parity dipole ( $l = 1$ ) mode and the even-parity monopole ( $l = 0$ ) mode. In these two special cases the dynamics is governed by a single Schrödinger-like wave equation, see Eqs. (7) and (16).
  - [14] Note that the monopole ( $l = 0$ ) mode does not exist in the odd-parity sector of spin-2 perturbations [5].
  - [15] Here the field  $\psi$  corresponds to the function  $Q$  in Eq. (36) of [5].
  - [16] Here the fields  $\psi_1$  and  $\psi_2$  respectively correspond to the functions  $Q$  and  $Z$  in Eqs. (32)-(33) of [5].
  - [17] Defining  $\phi \equiv \psi_1 + b \cdot \psi_2$ , one obtains from Eqs. (9)-(10) the wave equation:  $(\frac{d^2}{dr^2} + \omega^2 - \mu^2)\phi - \frac{l(l+1)+4+2b}{r^2}\{\psi_1 + \frac{(2+b)[l(l+1)-2]}{l(l+1)+4+2b}\psi_2\} = 0$ . Demanding the equality  $\frac{(2+b)[l(l+1)-2]}{l(l+1)+4+2b} = b$ , one obtains the quadratic equation  $b^2 + 3b - [l(l+1) - 2] = 0$  for the coefficient  $b$ , whose solutions are given by Eq. (11). The decoupled wave equations are then given by  $[\frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{l(l+1)+4+2b}{r^2}]\phi = 0$ , see Eqs. (12)-(13).
  - [18] Here the field  $\psi$  corresponds to the function  $\varphi_0$  in Eq. (30) of [5].
  - [19] Note that the flat-space wave equation (16) for the even-parity monopole ( $l = 0$ ) mode is identical to the flat-space wave equation (7) for the odd-parity dipole ( $l = 1$ ) mode.
  - [20] Here we have used the well known fact that the asymptotic late-time ( $t \rightarrow \infty$ ) tail is dominated by frequencies with  $|\omega| \rightarrow \mu$  [6, 7] in order to simplify the equations.
  - [21] Here the fields  $\psi_1$  and  $\psi_2$  respectively correspond to the functions  $rK$  and  $\eta_1$  in Eqs. (44)-(45) of [5].
  - [22] Defining  $\phi \equiv \psi_1 + b \cdot \psi_2$ , one obtains from Eqs. (18)-(19) the wave equation:  $(\frac{d^2}{dr^2} + \omega^2 - \mu^2)\phi - \frac{8-6b}{r^2}(\psi_1 + \frac{-4+6b}{8-6b}\psi_2) = 0$ . Demanding the equality  $\frac{-4+6b}{8-6b} = b$ , one obtains the quadratic equation  $3b^2 - b - 2 = 0$  for the coefficient  $b$ , whose solutions are given by Eq. (20). The decoupled wave equations are then given by  $(\frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{8-6b}{r^2})\phi = 0$ , see Eqs. (21)-(22).
  - [23] Here the fields  $\psi_1$ ,  $\psi_2$ , and  $\psi_3$  respectively correspond to the functions  $rK$ ,  $\eta_1$ , and  $rG$  in Eqs. (38)-(40) of [5].
  - [24] Defining  $\phi \equiv \psi_1 + b \cdot \psi_2 + c \cdot \psi_3$ , one obtains from Eqs. (25)-(27) the wave equation:  $(\frac{d^2}{dr^2} + \omega^2 - \mu^2)\phi - \frac{l(l+1)+6-6b}{r^2}\{\psi_1 + \frac{-2l(l+1)+b[l(l+1)+4]+2c}{l(l+1)+6-6b}\psi_2 + \frac{2b[l(l+1)-2]+c[l(l+1)-2]}{l(l+1)+6-6b}\psi_3\} = 0$ . Demanding the two equalities  $\frac{-2l(l+1)+b[l(l+1)+4]+2c}{l(l+1)+6-6b} = b$  and  $\frac{2b[l(l+1)-2]+c[l(l+1)-2]}{l(l+1)+6-6b} = c$ , one obtains the cubic equation  $9b^3 - 15b^2 + [6 - 4l(l+1)]b + 4l(l+1) = 0$  for the coefficient  $b$ , whose

solutions are given by Eqs. (28)-(30). The decoupled wave equations are then given by  $[\frac{d^2}{dr^2} + \omega^2 - \mu^2 - \frac{l(l+1)+6-6b}{r^2}]\phi = 0$ , see Eqs. (31)-(33).

- [25] It is worth emphasizing that this term disappears in the case of a flat spacetime with  $M \rightarrow 0$ , see Sec. II.
- [26] The coupling terms are of order  $O(1/r^2)$ , see Eqs. (9)-(10), (18)-(19), and (25)-(27). On the other hand, the curvature term is of order  $O(M\mu^2/r)$ , see Eq. (37). Thus, the curvature term dominates over the coupling terms in the asymptotic limit  $r \gg M/(M\mu)^2$ .
- [27] H. Koyama and A. Tomimatsu, Phys. Rev. D **64**, 044014 (2001); H. Koyama and A. Tomimatsu, Phys. Rev. D **65**, 084031 (2002).
- [28] The only exception is the even-parity monopole ( $l = 0$ ) mode. As shown in [5], this mode produces a Gregory-Laflamme-like instability of the Schwarzschild black hole in the regime  $M\mu \lesssim 0.43$ .